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Remark on Nonexistence of Global Solutions of the Initial-Boundary-Value Problem for the Nonlinear Klein–Gordon Equation

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Sufficient conditions are given so that the solutions of the initial-boundary-value problem for the nonlinear Klein–Gordon equation do not exist for all $t > 0$.

1. INTRODUCTION

Consider the initial-boundary-value problem (IBVP) for the nonlinear Klein–Gordon equation:

$$u_{tt} - \Delta u + \mu u = f(|u|^2)u, \qquad t \in [0, T), \qquad x \in \Omega, \qquad \Omega \subset \mathbf{R}^n$$

$$u(0, x) = u_0(x), \qquad x \in \Omega$$

$$u_t(0, x) = u_1(x), \qquad x \in \Omega$$

$$u(t, x)\big|_{x \in \partial\Omega} = 0, \qquad t \in [0, T)$$

The above problem has various applications in nonlinear optics (especially instability phenomena such as self-focusing), plasma physics, fluid mechanics, etc. We obtain some *a priori* estimates for the solutions of the IBVP under consideration. We give conditions on the initial functions u_0 and u_1 and on the function f such that the solution of the above problem blows up at a finite time $t = T$. The singularity of the solution occurs at $x = 0$ and is δ-function-like.

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2. PRELIMINARY NOTES

Let Ω be a bounded domain in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$ and $\{0, \ldots, 0\} \in \Omega$. We define $G = [0, T) \times \Omega$, $G_0 = [0, T) \times \bar{\Omega}$, where $T > 0$, $\bar{\Omega} = \Omega \cup \partial\Omega$.

Let us consider the IBVP for the nonlinear Klein–Gordon equation:

$$u_{tt} - \Delta u + \mu u = f(|u|^2)u \qquad \text{on } G \tag{1}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega \tag{2}$$

$$u_t(0, x) = u_1(x), \qquad x \in \Omega \tag{3}$$

$$u(t, x)\big|_{x \in \partial\Omega} = 0, \qquad t \in [0, T) \tag{4}$$

where $\mu \geq 0$ is a constant, f is a given real-valued function, and u_0, u_1 are given complex-valued functions.

We will consider the following Banach spaces of measurable functions with the norms:

$$L^q(\Omega) = \left\{ u(x)\colon \|u\|_{q,\Omega} = \left(\int_\Omega |u(x)|^q \, dx \right)^{1/q} < \infty \right\}$$

$$W_q^l(\Omega) = \left\{ u(x)\colon \|u\|_{W_q^l(\Omega)} = \sum_{j=0}^{l} \|D_x^j u\|_{q,\Omega} < \infty \right\}$$

$$\mathring{W}_q^l(\Omega) = W_q^l(\Omega) \cap \left\{ u(x)\colon u(x)\big|_{x \in \partial\Omega} = 0 \right\}$$

In the sequel we need the following theorem.

Theorem 1 (Ladyzhenskaya *et al.*, 1967, pp. 84–85). For each function $u \in \mathring{W}_2^1(\Omega)$ we have the inequality

$$\|u\|_{2,\Omega} \leq \beta(\text{mes } \Omega)^{1/n} \cdot \|\nabla u\|_{2,\Omega}$$

where

$$\beta = \begin{cases} \dfrac{2(n-1)}{(n-2)} & \text{if} \quad n \geq 3 \\[2ex] 2 & \text{if} \quad n = 1 \text{ or } n = 2 \end{cases}$$

We denote by $\bar{u}$ the complex conjugate of u.

3. MAIN RESULTS

First of all we obtain some *a priori* estimates for the solutions of the IBVP (1)–(4).

Lemma 1. Let $u \in C^2(G) \cap C^1(G_0)$ be a solution of the IBVP (1)–(4). Then

$$E(t) = C_0 + \int_\Omega F(|u(t, x)|^2)\, dx \tag{5}$$

where

$$E(t) = \|u_t(t)\|_{2,\Omega}^2 + \|\nabla u(t)\|_{2,\Omega}^2 + \mu\|u(t)\|_{2,\Omega}^2$$

$$C_0 = \|u_1\|_{2,\Omega}^2 + \|\nabla u_0\|_{2,\Omega}^2 + \mu\|u_0\|_{2,\Omega}^2 - \int_\Omega F(|u_0(x)|^2)\, dx$$

$$F(|u|^2) = \int_0^{|u|^2} f(s)\, ds$$

We omit the proof of Lemma 1.

Lemma 2. Let the following conditions hold:
1. $u \in C^2(G) \cap C^1(G_0)$ is a solution of the IBVP (1)–(4).

2. $$s \cdot f(s) - \int_0^s f(k)\, dk \geq 2M_1 s - M_2 \tag{6}$$

for $s \geq 0$, where

$$\frac{1}{\beta^2(\text{mes }\Omega)^{2/n}} + \mu \geq M_1 \geq \frac{1}{16} \qquad M_2 \geq 0$$

are given constants.

Then

$$\Gamma(t) \leq \int_\Omega F(|u(t, x)|^2)\, dx, \qquad t \in [0, T) \tag{7}$$

where

$$\Gamma(t) = \frac{1}{2}\{(C_1 - C_0')e^t + M_2(\text{mes }\Omega) - C_0\}$$

$$C_0' = C_0 + M_2(\text{mes }\Omega), \qquad C_1 = \text{Re}\left\{\int_\Omega u_1\, \overline{u_0}\, dx\right\}$$

Proof. Let $G_t = \{(\tau, x): \tau \in [0, t], x \in \Omega\}$, $t < T$. Multiplying both sides of (1) by $\bar{u}$ and then integrating over G_t, we obtain

$$\int_{G_t} (u_{tt}\bar{u} - \Delta u\bar{u} + \mu u\bar{u})\, dx\, d\tau$$

$$= \int_{G_t} f(|u|^2)|u|^2\, dx\, d\tau$$

$$\int_{G_t} \left(\frac{d}{dt}(u_t\bar{u}) - |u_t|^2 - \nabla \cdot (\nabla u\bar{u}) + |\nabla u|^2 + \mu|u|^2\right) dx\, d\tau$$

$$= \int_{G_t} f(|u|^2)|u|^2\, dx\, d\tau$$

$$\mathrm{Re}\left\{\int_\Omega u_t(t, x)\bar{u}(t, x)\, dx\right\} - \int_0^t \|u_t(\tau)\|_{2,\Omega}^2\, d\tau$$

$$+ \int_0^t \|\nabla u(\tau)\|_{2,\Omega}^2\, d\tau + \mu \int_0^t \|u(\tau)\|_{2,\Omega}^2\, d\tau$$

$$= \int_{G_t} f(|u(\tau, x)|^2)|u(\tau, x)|^2\, d\tau\, dx + C_1$$

On the other hand, (5) implies

$$\mathrm{Re}\left\{\int_\Omega u_t(t, x)\bar{u}(t, x)\, dx\right\}$$

$$= 2\int_0^t \|u_t(\tau)\|_{2,\Omega}^2\, d\tau + C_1 - C_0 t$$

$$+ \int_{G_t} f(|u(\tau, x)|^2)|u(\tau, x)|^2\, d\tau\, dx - \int_{G_t} F(|u(\tau, x)|^2)\, d\tau\, dx$$

Therefore, the inequality (6) yields

$$\left|\int_\Omega u_t(t, x)\bar{u}(t, x)\, dx\right|$$

$$\geq C_1 - C_0' t + 2\int_0^t (\|u_t(\tau)\|_{2,\Omega}^2 + M_1\|u(\tau)\|_{2,\Omega}^2)\, d\tau$$

Now we use Young's inequality in order to obtain

$$2\|u_t(t)\|_{2,\Omega}^2 + \frac{1}{8}\|u(t)\|_{2,\Omega}^2 \geq \left|\int_\Omega u_t(t, x)\bar{u}(t, x)\, dx\right|$$

Since $M_1 \geq 1/16$, we get

$$\begin{aligned} &2\|u_t(t)\|_{2,\Omega}^2 + 2M_1\|u(t)\|_{2,\Omega}^2 \\ &\quad \geq 2\int_0^t (\|u_t(\tau)\|_{2,\Omega}^2 + M_1\|u(\tau)\|_{2,\Omega}^2)\, d\tau - C_0't + C_1 \end{aligned} \tag{8}$$

Let us define now

$$X(t) = 2\|u_t(t)\|_{2,\Omega}^2 + 2M_1\|u(t)\|_{2,\Omega}^2$$

Then the inequality (8) has the form

$$X(t) \geq \int_0^t X(\tau)\, d\tau - C_0't + C_1$$

which is a Gronwall-type inequality.

Denoting

$$Y(t) = \int_0^t X(\tau)\, d\tau - C_0't + C_1$$

we obtain

$$Y'(t) = X(t) - C_0' \geq Y(t) - C_0'$$

$$Y(0) = C_1$$

Let

$$Z'(t) = Z(t) - C_0'$$

$$Z(0) = C_1$$

It is easy to prove that $Z(t) \leq Y(t)$ for $t \in [0, T)$. Therefore we conclude that

$$X(t) \geq Y(t) \geq Z(t) = (C_1 - C_0')e^t + C_0'$$

In other words,

$$2\|u_t(t)\|_{2,\Omega}^2 + 2M_1\|u(t)\|_{2,\Omega}^2 \geq (C_1 - C_0')e^t + C_0'$$

Now (5) and Theorem 1 imply

$$
\begin{aligned}
(C_1 - C_0')e^t + C_0' &\le 2C_0 + 2\int_\Omega F(|u(t,x)|^2)\,dx - 2\|\nabla u(t)\|_{2,\Omega}^2 \\
&\quad - 2\mu\|u(t)\|_{2,\Omega}^2 + 2M_1\|u(t)\|_{2,\Omega}^2 \\
&\le 2C_0 + 2\int_\Omega F(|u(t,x)|^2)\,dx
\end{aligned}
$$

Therefore, we have the inequality

$$\Gamma(t) \le \int_\Omega F(|u(t,x)|^2)\,dx, \qquad t \in [0, T) \quad \blacksquare$$

Theorem 2. Suppose that the following conditions are fulfilled:
1. The conditions of Lemma 2 hold.

2. $$|F(s)| \le \gamma \cdot s^p \tag{9}$$

where $s \ge 0$, $\gamma > 0$, $p > 1$.

3. $$\Gamma(T) > 0 \tag{10}$$

If

$$\lim_{\substack{t\to T\\ t<T}} \int_\Omega |x| \cdot |u|^{2p(1+1/n)}\,dx = 0$$

then

$$\lim_{\substack{t\to T\\ t<T}} \|u(t)\|_{q,\Omega} = 0 \qquad \text{for} \quad 1 \le q < 2p$$

$$\lim_{\substack{t\to T\\ t<T}} \|u(t)\|_{q,(|x|<\epsilon)} = \infty \qquad \text{for} \quad 2p < q \le \infty$$

for each fixed and sufficiently small $\epsilon > 0$.

Proof. By means of Lemma 2 and (9) we have the inequalities

$$\Gamma(t) \le \int_\Omega F(|u(t,x)|^2)\,dx \le \gamma \int_\Omega |u(t,x)|^{2p}\,dx$$

It follows from the Hölder inequality that

$$\int_{\Omega} |u|^{2p}\,dx = \int_{\Omega} |u|^{p}|u|^{p}\,dx$$
$$\le \left(\int_{\Omega} |u|^{ps}\,dx\right)^{1/s}\left(\int_{\Omega} |u|^{pq}\,dx\right)^{1/q}$$
$$= \|u\|^{p}_{sp,\Omega}\cdot\|u\|^{p}_{qp,\Omega}$$

where $s \ge 1$, $q \ge 1$, $1/s + 1/q = 1$. Therefore for fixed and sufficiently small $\epsilon > 0$ we have that

$$\Gamma(t) \le \gamma \int_{|x|<\epsilon} |u(t,x)|^{2p}\,dx + \gamma \int_{\substack{|x|>\epsilon \\ x\in\Omega}} |u(t,x)|^{2p}\,dx$$
$$\le \gamma\|u(t)\|^{p}_{sp,(|x|<\epsilon)}\cdot\|u(t)\|^{p}_{qp,(|x|<\epsilon)} + \gamma \int_{\substack{|x|>\epsilon \\ x\in\Omega}} |u(t,x)|^{2p}\,dx \qquad (11)$$

where $s \ge 1$, $q \ge 1$, $1/s + 1/q = 1$.

Now if $1 \le s < 2$, the Hölder inequality enable us to get

$$\|u\|^{2p(1+1/n)}_{sp,\Omega}$$
$$= \left(\int_{\Omega} |u|^{sp}\,dx\right)^{2(1+1/n)/s}$$
$$= \left(\int_{\Omega} |x|^{-s/2(1+1/n)}|x|^{s/2(1+1/n)}|u|^{sp}\,dx\right)^{2(1+1/n)/s}$$
$$\le \left(\int_{\Omega} |x|^{-\{[2(1+1/n)/s]-1\}^{-1}}\,dx\right)^{2(1+1/n)/s-1}$$
$$\times \left(\int_{\Omega} |x|\cdot|u|^{2p(1+1/n)}\,dx\right)$$
$$\le A\left(\int_{\Omega} |x|\cdot|u|^{2p(1+1/n)}\,dx\right) \to 0 \qquad \text{as} \quad t \to T, \quad t < T$$

Therefore

$$\lim_{\substack{t\to T \\ t<T}} \|u(t)\|_{q,\Omega} = 0 \qquad \text{if} \quad 1 \le q < 2p$$

It is not difficult to obtain

$$\left(\int_{\substack{|x|>\epsilon\\ x\in\Omega}} |u|^{2p}\,dx\right)^{1+1/n}$$

$$= \|u\|_{2p,(|x|>\epsilon;x\in\Omega)}^{2p(1+1/n)}$$

$$\leq B\|u\|_{2p(1+1/n),(|x|>\epsilon;x\in\Omega)}^{2p(1+1/n)}$$

$$= B\int_{\substack{|x|>\epsilon\\ x\in\Omega}} |u|^{2p(1+1/n)}\,dx$$

$$\leq \frac{B}{\epsilon}\int_{\substack{|x|>\epsilon\\ x\in\Omega}} |x|\cdot|u|^{2p(1+1/n)}\,dx$$

$$\leq \frac{B}{\epsilon}\int_{x\in\Omega} |x|\cdot|u|^{2p(1+1/n)}\,dx \to 0 \qquad \text{as} \quad t\to T, \quad t<T$$

It follows now from (11) that

$$\lim_{\substack{t\to T\\ t<T}} \|u(t)\|_{q,(|x|<\epsilon)} = \infty \qquad \text{if} \quad 2p<q\leq\infty \quad \blacksquare$$

Remark 1. Let $f(s) = s$ and $M_2 \geq 2M_1^2$. Then (6) is fulfilled.

Remark 2. Assume $f(s) = s^{\lambda-1}$, $\lambda > 1$, $s \geq 0$. Then $|F(s)| = (1/\lambda)\, s^{\lambda}$, $\lambda > 1$, and therefore (9) holds.

Remark 3. The inequality (10) deals with the initial functions u_0 and u_1. Let us consider the next example:

$$\Omega = [0, 1], \qquad T = \ln 2, \qquad M_2 = 5, \qquad \mu = 1$$

$$F(s) = 100\int_0^s k\,dk = 50s^2$$

$$u_0(x) = x^2 - x, \qquad u_1(x) = \frac{x^2 - x}{4}, \qquad x \in [0, 1]$$

Then $\Gamma(T) = \Gamma(\ln 2) > 0$.

Remark 4. The assumption

$$\lim_{\substack{t \to T \\ t < T}} \int_{\Omega} |x| \cdot |u|^{2p(1+1/n)}\, dx = 0$$

of Theorem 2 comes from an experimental point of view. The numerical computations show that the singularity of the solution occurs at $x = 0$ and is δ-function-like (Kelley, 1965; Zakharov *et al.*, 1971). The exact numerical computations of integrals of the type $\int_{\Omega} |u(t, x)|^p\, dx$ for $t \to T$, $t < T$, are difficult due to the presence of such a singularity of the solution. In contrast, integrals of the type $\int_{\Omega} |x| \cdot |u|^p\, dx$ can be calculated numerically with sufficient exactness for $t \to T$, $t < T$.

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